Permutation gates in the third level of the Clifford hierarchy

Zhiyang He*, Luke Robitaille*, and Xinyu Tan*

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA

The Clifford Hierarchy

- Let $\mathcal{C}_1 = \mathcal{P}$ be the Pauli group on n qubits, and inductively define
$$\mathcal{C}_k = \{U : \forall P \in \mathcal{P}, UPU^{-1} \in \mathcal{C}_{k-1}\}.$$
The Clifford Hierarchy is defined as $\mathcal{CH} := \cup_{k=1}^{\infty}\mathcal{C}_k$; we say $\mathcal{C}_k$ is its kth layer.
- $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3 \subseteq \cdots$ where $\mathcal{C}_1$ is the Pauli group, and $\mathcal{C}_2$ is the Clifford group. Non-Clifford gates are necessary for universal quantum computation!
- The Clifford hierarchy was introduced by Gottesman and Chuang [1] in 1999 in the context of gate teleportation, but has since been studied in its own right.
- For $k \geq 3$, $\mathcal{C}_k$ is not a group! For such a fundamental object in quantum computation, the structure of $\mathcal{CH}$ is not well understood.

What is known?

There has been lots of work aiming to understanding the structure of $\mathcal{CH}$ or $\mathcal{C}_3$.

- A semi-Clifford gate is $\phi_1 d \phi_2$ for Clifford gates ϕ_1, ϕ_2 and diagonal gate d.
- A generalized semi-Clifford gate is $\phi_1 \pi d \phi_2$ for Clifford gates ϕ_1, ϕ_2, permutation gate π, and diagonal gate d.
- Zeng, Chen, and Chuang [2] conjectured in 2007 that all elements of $\mathcal{C}_3$ are semi-Clifford, and all elements of $\mathcal{CH}$ are generalized semi-Clifford.
- Beigi and Shor [3] showed in 2008 that all elements of $\mathcal{C}_3$ are generalized semi-Clifford. We don't know if this is true for higher levels!
- Gottesman and Mochon [3] gave a non–semi-Clifford element of $\mathcal{C}_3$ on $n = 7$ qubits, as shown in Figure 2. Before our paper, this was the only known example of a non–semi-Clifford $\mathcal{C}_3$ gate!

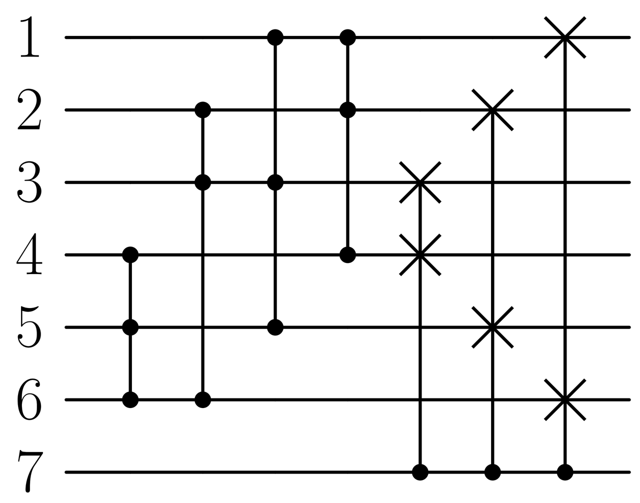

Figure 2: $\text{CSWAP}_{7,1,6}\text{CSWAP}_{7,2,5}\text{CSWAP}_{7,3,4} \cdot \text{CCZ}_{1,2,4}\text{CCZ}_{1,3,5}\text{CCZ}_{2,3,6}\text{CCZ}_{4,5,6}$.

Permutation gates in $\mathcal{C}_3$

- A permutation gate is a gate that permutes the computational basis states; in particular, there are $(2^n)!$ permutation gates on n qubits.
- For example, X gates and CNOT gates are permutation gates.
- A Toffoli gate acts on three qubits by
$$|a_1\rangle \otimes |a_2\rangle \otimes |a_3\rangle \mapsto |a_1\rangle \otimes |a_2\rangle \otimes |a_3 + a_1 a_2\rangle,$$
this sum $a_3 + a_1 a_2$ is considered to be over $\mathbb{F}_2$. We will refer to the first two qubits as controls and the third as the target.
- Let $\text{TOF}_{i,j,k}$ denote a Toffoli gate with qubits i and j as controls and qubit k as target.

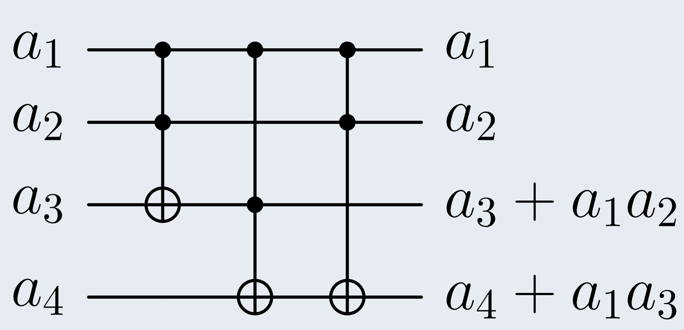

Figure 1: This circuit is for $\text{TOF}_{1,2,4}\text{TOF}_{1,3,4}\text{TOF}_{1,2,3}$.

- Permutation gates are very important for implementing circuits and algorithms, such as a quantum adder.

- In 2022, Anderson conjectured that all permutation gates can be written as a mismatch-free product of Toffoli gates, i.e. no qubit is ever used both as a control and as a target. For example, Figure 1 is not mismatch-free.
- Anderson's conjecture, if true, would imply that all permutation gates in $\mathcal{C}_3$ are semi-Clifford.
- This leads us to our first main result:

Any semi-Clifford permutation in $\mathcal{C}_3$ can be written, up to multiplying by Clifford permutations on both sides, as a mismatch-free product of Toffoli gates.

Permutation gates in $\mathcal{C}_3$ are staircases

- A product of Toffoli gates is in staircase form if the following two conditions hold:
  - each gate $\text{TOF}_{i,j,k}$ that appears has $i, j < k$, and
  - the target qubits are in nondecreasing order in the order that the gates are applied.

  For example, Figure 1 is in staircase form.
- Our second main result is the following:

Any permutation in $\mathcal{C}_3$ can be written, up to multiplying by Clifford permutations on both sides, as a product of Toffoli gates in staircase form.

- This is not an if-and-only-if; it is possible for a product of Toffoli gates in staircase form to not be in $\mathcal{C}_3$.

Non–semi-Clifford permutations in $\mathcal{C}_3$

- Our third main result rejects Anderson's conjecture:

Not all permutations in $\mathcal{C}_3$ are semi-Clifford. Example in Figure 3.

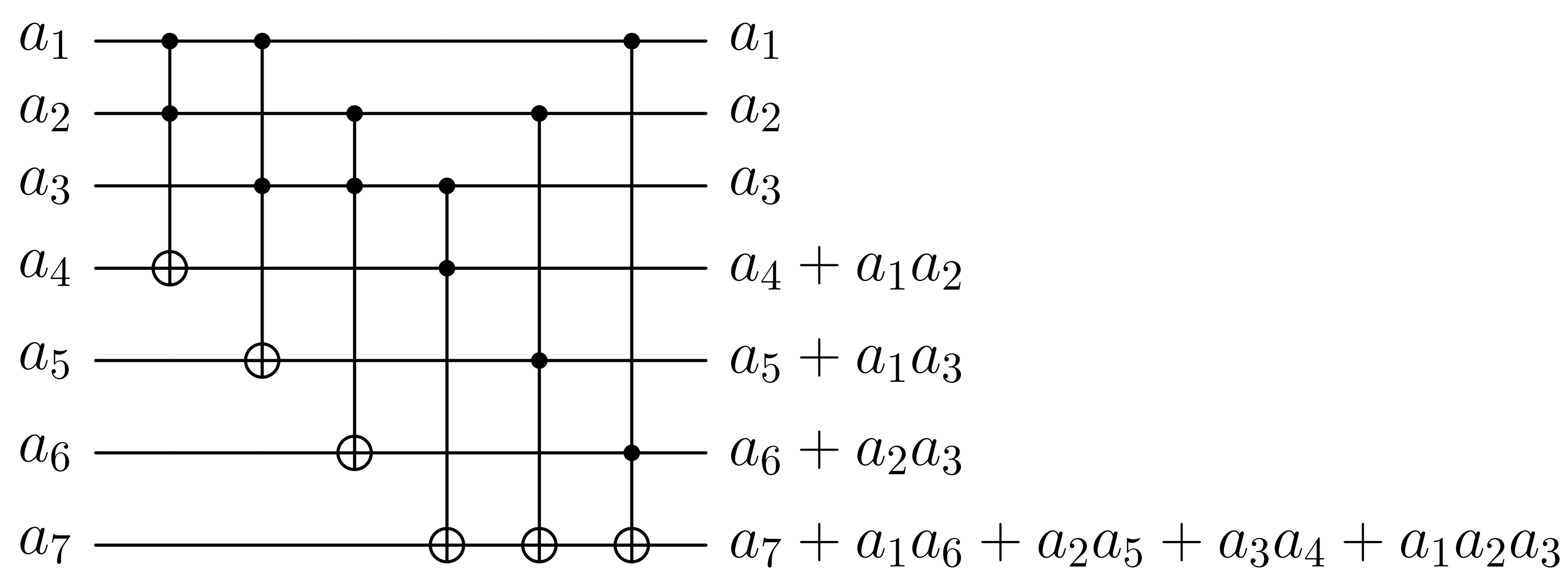

Figure 3: We show that the above gate, denoted as U_3, is in $\mathcal{C}_3$ but not semi-Clifford. In fact, this gate is conjugate to the gate in Figure 2 by a Clifford gate.

- Furthermore, using our staircase form characterization, we prove by a computer search that

$n = 7$ is the smallest number of qubits for which a non–semi-Clifford $\mathcal{C}_3$ permutation gate exists.

More Counterexamples

- The gate U_3 in Figure 3 is such that $U_3 \in \mathcal{C}_3$ but $U_3^{-1} \notin \mathcal{C}_3$. (This thereby disproves another conjecture of Anderson [4].)
- More generally, for any $k \geq 3$, we find a permutation gate U_k on $n = 2^k - 1$ qubits such that $U_k \in \mathcal{C}_3$ but $U_k^{-1} \notin \mathcal{C}_k$ (upcoming result).

Main Results

To summarize:

- Any permutation in $\mathcal{C}_3$ can be written, up to multiplying by Clifford permutations on both sides, as a product of Toffoli gates in staircase form.
- The smallest number of qubits for which there exists a non–semi-Clifford permutation in $\mathcal{C}_3$ is $n = 7$.
- Any semi-Clifford permutation in $\mathcal{C}_3$ can be written, up to multiplying by Clifford permutations on both sides, as a mismatch-free product of Toffoli gates.

References

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